

A Necessary and Sufficient Condition for the Continuity of Local Minima of Parabolic Variational Integrals with Linear Growth

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Abstract

For proper minimizers of parabolic variational integrals with linear growth with respect to $|Du|$, we establish a necessary and sufficient condition for u to be continuous at a point (x_o, t_o) , in terms of a sufficient fast decay of the total variation of u about (x_o, t_o) (see (1.4) below). These minimizers arise also as proper solutions to the parabolic 1-laplacian equation. Hence, the continuity condition continues to hold for such solutions (§ 3).

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1 Introduction

Let E be an open subset of \mathbb{R}^N , and denote by $BV(E)$ the space of functions $v \in L^1(E)$ with finite total variation [9]

$$\|Dv\|(E) := \sup_{\substack{\varphi \in [C_o^1(E)]^N \\ |\varphi| \leq 1}} \left\{ \langle Dv, \varphi \rangle = - \int_E v \operatorname{div} \varphi \, dx \right\} < \infty.$$

Here $Dv = (D_1v, \dots, D_Nv)$ is the vector valued Radon measure, representing the distributional gradient of v . A function $v \in BV_{\text{loc}}(E)$ if $v \in BV(E')$ for all open sets $E' \subsetneq E$. For $T > 0$, let $E_T = E \times (0, T)$, and denote by $L^1(0, T; BV(E))$ the collection of all maps $v : [0, T] \rightarrow BV(E)$ such that

$$v \in L^1(E_T), \quad \|Dv(t)\|(E) \in L^1(0, T),$$

and the maps

$$(0, T) \ni t \rightarrow \langle Dv(t), \varphi \rangle$$

are measurable with respect to the Lebesgue measure in \mathbb{R} , for all $\varphi \in [C_o^1(E)]^N$.

A function $u \in L_{\text{loc}}^1(0, T; BV_{\text{loc}}(E))$ is a local parabolic minimizer of the total variation flow in E_T , if

$$\int_0^T \left[\int_E -u \varphi_t \, dx + \|Du(t)\|(E) \right] dt \leq \int_0^T \|D(u + \varphi)(t)\|(E) dt \quad (1.1)$$

for all non-negative $\varphi \in C_o^\infty(E_T)$. The notion has been introduced in [3]. It is a parabolic version of the elliptic local minima of total variation flow as introduced in [10].

1.1 The Main Result

Let $B_\rho(x_o)$ denote the ball of radius ρ about x_o . If $x_o = 0$, write $B_\rho(x_o) = B_\rho$. Introduce the cylinders $Q_\rho(\theta) = B_\rho \times (-\theta\rho, 0]$, where θ is a positive parameter to be chosen as needed. If $\theta = 1$ we write $Q_\rho(1) = Q_\rho$. For a point $(x_o, t_o) \in \mathbb{R}^{N+1}$ we let $[(x_o, t_o) + Q_\rho(\theta)]$ be the cylinder of “vertex” at (x_o, t_o) and congruent to $Q_\rho(\theta)$, i.e.,

$$[(x_o, t_o) + Q_\rho(\theta)] = B_\rho(x_o) \times (t_o - \theta\rho, t_o],$$

and we let $\rho > 0$ be so small that $[(x_o, t_o) + Q_\rho(\theta)] \subset E_T$.

Theorem 1.1 *Let $u \in L_{\text{loc}}^1(0, T; BV_{\text{loc}}(E))$ be a local parabolic minimizer of the total variation flow in E_T , satisfying in addition*

$$u \in L_{\text{loc}}^\infty(E_T) \quad \text{and} \quad u_t \in L_{\text{loc}}^1(E_T). \quad (1.2)$$

Then, u is continuous at some $(x_o, t_o) \in E_T$, if and only if

$$\limsup_{\rho \searrow 0} \frac{\rho}{|Q_\rho|} \int_{t_o - \rho}^{t_o} \|Du(\cdot, t)\|(B_\rho(x_o)) \, dt = 0. \quad (1.3)$$

For stationary, elliptic minimizers, condition (1.3) has been introduced in [10]. The stationary version of (1.3) implies that u is quasi-continuous at x_o . For time-dependent minimizers, however, (1.3) gives no information on the possible quasi-continuity of u at (x_o, t_o) . Condition (1.3), is only a measure-theoretical restriction on the speed at which a possible discontinuity may develop at (x_o, t_o) . For this reason our proof is entirely different than [10], being based instead on a DeGiorgi-type iteration technique that exploits precisely such a measure-theoretical information.

2 Comments on Boundedness and Continuity

The theorem requires that u is locally bounded and that $u_t \in L^1_{\text{loc}}(E_T)$. In the elliptic case, local minimizers of the total gradient flow in E , are locally bounded ([10, § 2]). This is not the case, in general, for parabolic minimizers in E_T , even if $u_t \in C^\infty_{\text{loc}}(0, T; L^1_{\text{loc}}(E))$. Consider the function

$$B_1 \times (-\infty, 1) \ni (x, t) \rightarrow F(|x|, t) = (1 - t) \frac{N - 1}{|x|}, \quad \text{for } N \geq 3.$$

Denote by $D_a F$ that component of the measure DF which is absolutely continuous with respect to the Lebesgue measure in \mathbb{R}^N . One verifies that $DF = D_a F$ and $\|DF(t)\|(B_1) = \|D_a F(t)\|_{1, B_1}$. By direct computation

$$\int_0^T \int_{B_1} \left(-F\varphi_t + \frac{D_a F}{|D_a F|} \cdot D\varphi \right) dx dt = 0,$$

for all $\varphi \in C^\infty_o(B_1 \times (0, T))$, $0 < T < 1$. From this

$$\int_0^T \int_{B_1} \left(-F\varphi_t + \frac{D_a F}{|D_a F|} \cdot D_a F \right) dx dt = \int_0^T \int_{B_1} \frac{D_a F}{|D_a F|} \cdot D_a (F - \varphi) dx dt,$$

which yields

$$\int_0^T \int_{B_1} \left(-F\varphi_t + |D_a F| \right) dx dt \leq \int_0^T \int_{B_1} |D_a (F - \varphi)| dx dt.$$

Thus F is a local, unbounded, parabolic minimizer of the total variation flow. The requirement $u \in L^\infty_{\text{loc}}(E_T)$ could be replaced by asking that $u \in L^r_{\text{loc}}(E_T)$ for some $r > N$. A discussion on this issue is provided in Appendix B.

2.1 On the Modulus of Continuity

While Theorem 1.1 gives a necessary and sufficient condition for continuity at a given point, it provides no information on the modulus of continuity of u at (x_o, t_o) . Consider the two time-independent functions in $B_\rho \times (0, \infty)$, for some

$\rho < 1$:

$$u_1(x_1, x_2) = \begin{cases} \frac{1}{\ln x_1} & \text{for } x_1 > 0; \\ 0 & \text{for } x_1 = 0; \\ -\frac{1}{\ln(-x_1)} & \text{for } x_1 < 0. \end{cases}$$

$$u_2(x_1, x_2) = \begin{cases} \sqrt{x_1} & \text{for } x_1 > 0; \\ -\sqrt{-x_1} & \text{for } x_1 \leq 0. \end{cases}$$

Both are stationary parabolic minimizers of the total variation flow in the sense of (1.1)–(1.2), over $B_{\frac{1}{2}} \times (0, \infty)$. We establish this for u_1 , the analogous statement for u_2 being analogous. Since $u_1 \in W^{1,1}(B_\rho)$, and is time-independent, one also has $u \in L^1(0, T; BV(B_\rho))$. To verify (1.1), one needs to show that

$$\|Du_1\|(B_\rho) \leq \frac{1}{T} \int_0^T \|D(u_1 + \varphi)(\cdot, t)\|(B_\rho) dt \quad (*)$$

for all $T > 0$, and all $\varphi \in C_o^\infty(B_\rho \times (0, T))$. Let $\mathcal{H}^k(A)$ denote the k -dimensional Hausdorff measure of a Borel set $A \subset \mathbb{R}^N$. One checks that $\mathcal{H}^N([Du_1 = 0]) = 0$ and there exists a closed set $K \subset B_\rho$, such that $\mathcal{H}^{N-1}(K) = 0$ and

$$\int_{B_\rho - K} \frac{Du_1}{|Du_1|} \cdot D\varphi \, dx = 0, \quad \text{for all } \varphi \in C_o^\infty(B_\rho - K).$$

From this, by Lemma 4 of [5, § 8], for all $\psi \in C_o^\infty(B_\rho)$, one has

$$\|Du_1\|(B_\rho) \leq \|D(u_1 + \psi)\|(B_\rho),$$

which, in turn, yields (*). The two functions u_1 and u_2 can be regarded as equibounded near the origin. They both satisfy (1.3), and exhibit quite different moduli of continuity at the origin. This occurrence is in line with a remark of Evans ([8]). A sufficiently smooth minimizer of the elliptic functional $\|Du\|(E)$ is a function whose level sets are surfaces of zero mean curvature. Thus, if u is a minimizer, so is $\varphi(u)$ for all continuous monotone functions $\varphi(\cdot)$. This implies that a modulus of continuity cannot be identified solely in terms of an upper bound of u .

3 Singular Parabolic DeGiorgi Classes

Let $\mathcal{C}(Q_\rho(\theta))$ denote the class of all non-negative, piecewise smooth, cutoff functions ζ defined in $Q_\rho(\theta)$, vanishing outside B_ρ , such that $\zeta_t \geq 0$ and satisfying

$$|D\zeta| + \zeta_t \in L^\infty(Q_\rho(\theta)).$$

For a measurable function $u : E_T \rightarrow \mathbb{R}$ and $k \in \mathbb{R}$ set

$$(u - k)_\pm = \{\pm(u - k) \wedge 0\}.$$

The singular, parabolic DeGiorgi class $[DG]^\pm(E_T; \gamma)$ is the collection of all measurable maps

$$u \in C_{\text{loc}}((0, T); L_{\text{loc}}^2(E)) \cap L_{\text{loc}}^1(0, T; BV_{\text{loc}}(E)), \quad (3.1)$$

satisfying

$$\begin{aligned} & \sup_{t_o - \theta\rho \leq t \leq t_o} \int_{B_\rho(x_o)} (u - k)_\pm^2 \zeta(x, t) dx \\ & \quad + \int_{t_o - \theta\rho}^{t_o} \|D((u - k)_\pm \zeta)(\tau)\| (B_\rho(x_o)) dt \\ & \leq \gamma \iint_{[(x_o, t_o) + Q_\rho(\theta)]} [(u - k)_\pm |D\zeta| + (u - k)_\pm^2 |\zeta_t|] dx dt + \\ & \quad + \int_{B_\rho(x_o)} (u - k)_\pm^2 \zeta(x, t_o - \theta\rho) dx \end{aligned} \quad (3.2)$$

for all $[(x_o, t_o) + Q_\rho(\theta)] \subset E_T$, all $k \in \mathbb{R}$, and all $\zeta \in \mathcal{C}([(x_o, t_o) + Q_\rho(\theta)])$, for a given positive constant γ . The singular DeGiorgi classes $[DG](E_T; \gamma)$ are defined as $[DG](E_T; \gamma) = [DG]^+(E_T; \gamma) \cap [DG]^-(E_T; \gamma)$.

3.1 The Main Result

The main result of this note is that the necessary and sufficient condition of Theorem 1.1 holds for functions $u \in DG(E_T; \gamma) \cap L_{\text{loc}}^\infty(E_T)$. Indeed, the proof of Theorem 1.1, only uses the local integral inequalities (3.2). In particular, the second of (1.2) is not needed.

Proposition 3.1 *Let u in the functional classes (3.1), be a parabolic minimizer of the total variation flow in E_T , in the sense of (1.1), satisfying in addition (1.2). Then $u \in DG(E_T; 2)$.*

The proof will be given in Appendix A.

Remark 3.1 Note that in the context of $DG(E_T)$ classes, the characteristic condition (1.3), holds with no further requirement that $u_t \in L_{\text{loc}}^1(E_T)$. The latter however is needed to cast a parabolic minimizer of the total variation flow into a $DG(E_T)$ -class as stated by Proposition 3.1.

4 A Singular Diffusion Equation

Consider formally, the parabolic 1-Laplacian equation

$$u_t - \operatorname{div} \left(\frac{Du}{|Du|} \right) = 0 \quad \text{formally in } E_T. \quad (4.1)$$

Let \mathcal{P} be the class of all Lipschitz continuous, non-decreasing functions $p(\cdot)$ defined in \mathbb{R} , with p' compactly supported. Denote by $\mathcal{C}(E_T)$ the class of all non-negative functions ζ defined in E_T , such that $\zeta(\cdot, t) \in C_o^1(E)$ for all $t \in (0, T)$, and $0 \leq \zeta_t < \infty$ in E_T . A function $u \in C_{\text{loc}}(0, T; L^1(E))$ is a local solution to (4.1) if

- a. $p(u) \in L_{\text{loc}}^1(0, T; BV(E))$, for all $p \in \mathcal{P}$;
- b. there exists a vector valued function $\mathbf{z} \in [L^\infty(E_T)]^N$ with $\|\mathbf{z}\|_{\infty, E} \leq 1$, such that $u_t = \text{div } \mathbf{z}$ in $\mathcal{D}'(E_T)$;
- c. denoting by $d(\|Dp(u - \ell)\|)$ the measure in E generated by the total variation $\|Dp(u - \ell)\|(E)$

$$\begin{aligned} & \int_E \left(\int_0^{u-\ell} p(s) ds \right) \zeta(x, t_2) dx + \int_{t_1}^{t_2} \int_E \zeta d(\|D(p(u - \ell))\|) dt \\ & \leq \int_E \left(\int_0^{u-\ell} p(s) ds \right) \zeta(x, t_1) dx - \int_{t_1}^{t_2} \int_E \left(\int_0^{u-\ell} p(s) ds \right) \zeta_t dx dt \quad (4.2) \\ & \quad - \int_{t_1}^{t_2} \int_E \mathbf{z} \cdot D\zeta p(u - \ell) dx dt \end{aligned}$$

for all $\ell \in \mathbb{R}$, all $p \in \mathcal{P}$, all $\zeta \in \mathcal{C}(E_T)$ and all $[t_1, t_2] \subset (0, T)$. The notion is a local version of a global one introduced in [1, Chapter 3]. Similar notions are in [1, 3, 4, 11], associated with issues of existence for the Cauchy problem and boundary value problems associated with (4.1). The notion of solution in [3], called *variational*, is different and closely related to the variational integrals (1.1).

Our results are local in nature and disengaged from any initial or boundary conditions. Let u be a local solution to (4.1) in the indicated sense, which in addition is locally bounded in E_T . In (4.2) take $\ell = 0$, and $p_\pm(u) = \pm(u - k)_\pm$. Since $u \in L_{\text{loc}}^\infty(E_T)$ one verifies that $p_\pm \in \mathcal{P}$. Standard calculations then yield that u is in the DeGiorgi classes $[DG]^\pm(E; \gamma)$, for some fixed $\gamma > 0$. As a consequence, we have the following:

Corollary 4.1 *Let $u \in L_{\text{loc}}^\infty(E_T)$ be a local solution to (4.1), in E_T , in the sense (a)-(c) above. Then, u is continuous at some $(x_o, t_o) \in E_T$, if and only if (1.3) holds true.*

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5 Proof of the Necessary Condition

Let $u \in [DG](E_T; \gamma)$ be continuous at $(x_o, t_o) \in E_T$, which we may take as the origin of \mathbb{R}^{N+1} , and may assume $u(0, 0) = 0$. In $(3.2)_+$ for $(u - k)_+$, take $\theta = 1$

and $k = 0$. Let also $\zeta \in \mathcal{C}(Q_{2\rho})$ be such that $\zeta(\cdot, -2\rho) = 0$, such that $\zeta = 1$ on $Q_{\frac{3}{2}\rho}$, and

$$|D\zeta| + \zeta_t \leq \frac{3}{\rho}.$$

Repeat the same choices in (3.2)₋ for $(u - k)_-$. Adding the resulting inequalities gives

$$\frac{\rho}{|Q_\rho|} \int_{-2\rho}^0 \|D(u\zeta)(\cdot, t)\|(B_{2\rho}) dt \leq 2^{N+1} \gamma \iint_{Q_{2\rho}} (u + u^2) dx dt. \quad (5.1)$$

Since the total variation $\|Dw\|$ of a function $w \in BV$ can be seen as a measure (see, for example, [13, Chapter 1, § 1]), we have

$$\frac{\rho}{|Q_\rho|} \int_{-\rho}^0 \|D(u\zeta)(\cdot, t)\|(B_\rho) dt \leq \frac{\rho}{|Q_\rho|} \int_{-2\rho}^0 \|D(u\zeta)(\cdot, t)\|(B_{2\rho}) dt;$$

on the other hand, $u\zeta \equiv u$ in $Q_{\frac{3}{2}\rho} \supset Q_\rho$, and therefore we conclude

$$\frac{\rho}{|Q_\rho|} \int_{-\rho}^0 \|Du(\cdot, t)\|(B_\rho) dt \leq 2^{N+1} \gamma \iint_{Q_{2\rho}} (u + u^2) dx dt.$$

The right-hand side tends to zero as $\rho \rightarrow 0$, thereby implying the necessary condition of Theorem 1.1. \blacksquare

6 A DeGiorgi-Type Lemma

For a fixed cylinder $[(y, s) + Q_{2\rho}(\theta)] \subset E_T$, denote by μ_\pm and ω , non-negative numbers such that

$$\mu_+ \geq \operatorname{ess\,sup}_{[(y,s)+Q_{2\rho}(\theta)]} u, \quad \mu_- \leq \operatorname{ess\,inf}_{[(y,s)+Q_{2\rho}(\theta)]} u, \quad \omega \geq \mu_+ - \mu_-. \quad (6.1)$$

Let $\xi \in (0, \frac{1}{2}]$ be fixed and let $\theta = 2\xi\omega$. This is an intrinsic cylinder in that its length $\theta\rho$ depends on the oscillation of u within it. We assume momentarily that the indicated choice of parameters can be effected.

Lemma 6.1 *Let u belong to $[DG]^-(E_T, \gamma)$. There exists a number ν_- depending on N , and γ only, such that if*

$$|[u \leq \mu_- + \xi\omega] \cap [(y, s) + Q_{2\rho}(\theta)]| \leq \nu_- |Q_{2\rho}(\theta)|, \quad (6.2)$$

then

$$u \geq \mu_- + \frac{1}{2}\xi\omega \quad \text{a.e. in } [(y, s) + Q_\rho(\theta)]. \quad (6.3)$$

Likewise, if u belongs to $[DG]^+(E_T, \gamma)$, there exists a number ν_+ depending on N , and γ only, such that if

$$|[u \geq \mu_+ - \xi\omega] \cap [(y, s) + Q_{2\rho}(\theta)]| \leq \nu_+ |Q_{2\rho}(\theta)|, \quad (6.4)$$

then

$$u \leq \mu_+ - \frac{1}{2}\xi\omega \quad \text{a.e. in } [(y, s) + Q_\rho(\theta)]. \quad (6.5)$$

Proof: We prove (6.2)–(6.3), the proof for (6.4)–(6.5) being similar. We may assume $(y, s) = (0, 0)$ and for $n = 0, 1, \dots$, set

$$\rho_n = \rho + \frac{\rho}{2^n}, \quad B_n = B_{\rho_n}, \quad Q_n = B_n \times (-\theta\rho_n, 0].$$

Apply (3.2)_− over B_n and Q_n to $(u - k_n)_-$, for the levels

$$k_n = \mu_- + \xi_n \omega \quad \text{where} \quad \xi_n = \frac{1}{2}\xi + \frac{1}{2^{n+1}}\xi.$$

The cutoff function ζ is taken of the form $\zeta(x, t) = \zeta_1(x)\zeta_2(t)$, where

$$\begin{aligned} \zeta_1 &= \begin{cases} 1 & \text{in } B_{n+1} \\ 0 & \text{in } \mathbb{R}^N - B_n \end{cases} & |D\zeta_1| &\leq \frac{1}{\rho_n - \rho_{n+1}} = \frac{2^{n+1}}{\rho} \\ \zeta_2 &= \begin{cases} 0 & \text{for } t < -\theta\rho_n \\ 1 & \text{for } t \geq -\theta\rho_{n+1} \end{cases} & 0 \leq \zeta_{2,t} &\leq \frac{1}{\theta(\rho_n - \rho_{n+1})} = \frac{2^{(n+1)}}{\theta\rho}. \end{aligned}$$

Inequality (3.2)_− with these stipulations yields

$$\begin{aligned} & \operatorname{ess\,sup}_{-\theta\rho_n < t < 0} \int_{B_n} (u - k_n)_-^2 \zeta(x, t) dx + \int_{-\theta\rho_n}^0 \|D(u - k_n)_-\zeta\|(B_n) dt \\ & \leq \gamma \frac{2^n}{\rho} \left(\iint_{Q_n} (u - k_n)_- dx dt + \frac{1}{\theta} \iint_{Q_n} (u - k_n)_-^2 dx dt \right) \\ & \leq \gamma \frac{2^n(\xi\omega)}{\rho} |[u < k_n] \cap Q_n|. \end{aligned}$$

By the embedding Proposition 4.1 of [7, Preliminaries]

$$\begin{aligned} & \iint_{Q_n} [(u - k_n)_-\zeta]^{\frac{N+2}{N}} dx dt \leq \int_{-\theta\rho_n}^0 \|D[(u - k_n)_-\zeta]\|(B_n) dt \\ & \quad \times \left(\operatorname{ess\,sup}_{-\theta\rho_n < t < 0} \int_{B_n} [(u - k_n)_-\zeta(x, t)]^2 dx \right)^{\frac{1}{N}} \\ & \leq \gamma \left(\frac{2^n}{\rho} \xi\omega \right)^{\frac{N+1}{N}} |[u < k_n] \cap Q_n|^{\frac{N+1}{N}}. \end{aligned}$$

Estimate below

$$\iint_{Q_n} [(u - k_n)_-\zeta]^{\frac{N+2}{N}} dx dt \geq \left(\frac{\xi\omega}{2^{n+2}} \right)^{\frac{N+2}{N}} |[u < k_{n+1}] \cap Q_{n+1}|$$

and set

$$Y_n = \frac{|[u < k_n] \cap Q_n|}{|Q_n|}.$$

Then

$$Y_{n+1} \leq \gamma b^n Y_n^{1+\frac{1}{N}}$$

where

$$b = 2^{\frac{1}{N}[3N+4]}.$$

By Lemma 5.1 of [7, Preliminaries], $\{Y_n\} \rightarrow 0$ as $n \rightarrow \infty$, provided

$$Y_o \leq \gamma^{-N} b^{-N^2} \stackrel{\text{def}}{=} \nu_-.$$

The proof of (6.4)–(6.5) is almost identical. One starts from inequalities (3.2)₊ written for the truncated functions

$$(u - k_n)_+ \quad \text{with} \quad k_n = \mu_+ - \xi_n \omega$$

for the same choice of ξ_n . ■

7 A Time Expansion of Positivity

For a fixed cylinder

$$[(y, s) + Q_{2\rho}^+(\theta)] = B_{2\rho}(y) \times (s, s + \theta\rho) \subset E_T,$$

denote by μ_{\pm} and ω , non-negative numbers satisfying the analog of (6.1). Let also $\xi \in (0, 1)$ be a fixed parameter. The value of θ will be determined by the proof; we momentarily assume that such a choice can be done.

Lemma 7.1 *Let $u \in [DG]^-(E_T, \gamma)$ and assume that for some $(y, s) \in E_T$ and some $\rho > 0$*

$$|[u(\cdot, s) \geq \mu_- + \xi\omega] \cap B_{\rho}(y)| \geq \frac{1}{2}|B_{\rho}(y)|.$$

Then, there exist δ and ϵ in $(0, 1)$, depending only on N , γ , and independent of ξ , such that

$$|[u(\cdot, t) > \mu_- + \epsilon\xi\omega] \cap B_{\rho}(y)| \geq \frac{1}{4}|B_{\rho}| \quad \text{for all } t \in (s, s + \delta(\xi\omega)\rho].$$

Proof: Assume $(y, s) = (0, 0)$ and for $k > 0$ and $t > 0$ set

$$A_{k,\rho}(t) = [u(\cdot, t) < k] \cap B_{\rho}.$$

The assumption implies

$$|A_{\mu_- + \xi\omega, \rho}(0)| \leq \frac{1}{2}|B_{\rho}|. \tag{7.1}$$

Write down inequalities (3.2)_− for the truncated functions $(u - (\mu_- + \xi\omega))_-$, over the cylinder $B_{\rho} \times (0, \theta\rho]$, where $\theta > 0$ is to be chosen. The cutoff function ζ is taken independent of t , non-negative, and such that

$$\zeta = 1 \quad \text{on } B_{(1-\sigma)\rho}, \quad \text{and} \quad |D\zeta| \leq \frac{1}{\sigma\rho},$$

where $\sigma \in (0, 1)$ is to be chosen. Discarding the non-negative term containing $D(u - (\mu_- + \xi\omega))_-$ on the left-hand side, these inequalities yield

$$\begin{aligned} \int_{B_{(1-\sigma)\rho}} (u - (\mu_- + \xi\omega))_-^2(x, t) dx &\leq \int_{B_\rho} (u - (\mu_- + \xi\omega))_-^2(x, 0) dx \\ &\quad + \frac{\gamma}{\sigma\rho} \int_0^{\theta\rho} \int_{B_\rho} (u - (\mu_- + \xi\omega))_- dx dt \\ &\leq (\xi\omega)^2 \left[\frac{1}{2} + \gamma \frac{\theta}{\sigma(\xi\omega)} \right] |B_\rho| \end{aligned}$$

for all $t \in (0, \theta\rho]$, where we have enforced (7.1). The left-hand side is estimated below by

$$\begin{aligned} &\int_{B_{(1-\sigma)\rho}} (u - (\mu_- + \xi\omega))_-^2(x, t) dx \\ &\geq \int_{B_{(1-\sigma)\rho} \cap [u < \mu_- + \epsilon\xi\omega]} (u - (\mu_- + \xi\omega))_-^2(x, t) dx \\ &\geq (\xi\omega)^2 (1 - \epsilon)^2 |A_{\mu_- + \epsilon\xi\omega, (1-\sigma)\rho}(t)| \end{aligned}$$

where $\epsilon \in (0, 1)$ is to be chosen. Next, estimate

$$\begin{aligned} |A_{\mu_- + \epsilon\xi\omega, \rho}(t)| &= |A_{\mu_- + \epsilon\xi\omega, (1-\sigma)\rho}(t) \cup (A_{\mu_- + \epsilon\xi\omega, \rho}(t) - A_{\mu_- + \epsilon\xi\omega, (1-\sigma)\rho}(t))| \\ &\leq |A_{\mu_- + \epsilon\xi\omega, (1-\sigma)\rho}(t)| + |B_\rho - B_{(1-\sigma)\rho}| \\ &\leq |A_{\mu_- + \epsilon\xi\omega, (1-\sigma)\rho}(t)| + N\sigma|B_\rho|. \end{aligned}$$

Combining these estimates gives

$$\begin{aligned} |A_{\mu_- + \epsilon\xi\omega, \rho}(t)| &\leq \frac{1}{(\xi\omega)^2 (1 - \epsilon)^2} \int_{B_{(1-\sigma)\rho}} (u - (\mu_- + \xi\omega))_-^2(x, t) dx + N\sigma|B_\rho| \\ &\leq \frac{1}{(1 - \epsilon)^2} \left[\frac{1}{2} + \frac{\gamma\theta}{\sigma(\xi\omega)} + N\sigma \right] |B_\rho|. \end{aligned}$$

Choose $\theta = \delta(\xi\omega)$ and then set

$$\sigma = \frac{1}{16N}, \quad \epsilon \leq \frac{1}{32}, \quad \delta = \frac{1}{2^8 \gamma N}. \quad (7.2)$$

This proves the lemma. ■

8 Proof of the Sufficient Part of Theorem 1.1

Having fixed $(x_o, t_o) \in E_T$ assume it coincides with the origin of \mathbb{R}^{N+1} and let $\rho > 0$ be so small that $Q_\rho \subset E_T$. Set

$$\mu_+ = \operatorname{ess\,sup}_{Q_\rho} u, \quad \mu_- = \operatorname{ess\,inf}_{Q_\rho} u, \quad \omega = \mu_+ - \mu_- = \operatorname{ess\,osc}_{Q_\rho} u.$$

Without loss of generality, we may assume that $\omega \leq 1$ so that

$$Q_\rho(\omega) = B_\rho \times (-\omega\rho, 0] \subset Q_\rho \subset E_T$$

and

$$\operatorname{ess\,osc}_{Q_\rho(\omega)} u \leq \omega.$$

If u were not continuous at (x_o, t_o) , there would exist $\rho_o > 0$ and $\omega_o > 0$, such that

$$\omega_\rho = \operatorname{ess\,osc}_{Q_\rho} u \geq \omega_o > 0 \quad \text{for all } \rho \leq \rho_o. \quad (8.1)$$

Let δ be determined from the last of (7.2). At the time level $t = -\delta\omega\rho$, either

$$\begin{aligned} & |[u(\cdot, -\delta\omega\rho) \geq \mu_- + \tfrac{1}{2}\omega] \cap B_\rho| \geq \tfrac{1}{2}|B_\rho|, \quad \text{or} \\ & |[u(\cdot, -\delta\omega\rho) \leq \mu_+ - \tfrac{1}{2}\omega] \cap B_\rho| \geq \tfrac{1}{2}|B_\rho|. \end{aligned}$$

Assuming the former holds, by Lemma 7.1

$$|[u(\cdot, t) > \mu_- + \tfrac{1}{64}\omega] \cap B_\rho| \geq \tfrac{1}{4}|B_\rho| \quad \text{for all } t \in (-\delta\omega\rho, 0].$$

Let $2\xi = \frac{1}{64}\delta$. Then

$$|[u(\cdot, t) > \mu_- + 2\xi\omega] \cap B_\rho| \geq \tfrac{1}{4}|B_\rho| \quad \text{for all } t \in (-\xi\omega\rho, 0]. \quad (8.2)$$

Next, apply the discrete isoperimetric inequality of Lemma 2.2 of [7, Preliminaries] to the function $u(\cdot, t)$, for t in the range $(-\xi\omega\rho, 0]$, over the ball B_ρ , for the levels

$$k = \mu_- + \xi\omega \quad \text{and} \quad \ell = \mu_- + 2\xi\omega \quad \text{so that} \quad \ell - k = \xi\omega.$$

This inequality is stated and proved in [7] for functions in $W_{\text{loc}}^{1,1}(E)$. It continues to hold for $u \in BV_{\text{loc}}(E)$, by virtue of the approximation procedure of [9, Theorem 1.17]. Taking also into account (8.2) this gives

$$\xi\omega |[u(\cdot, t) < \mu_- + \xi\omega] \cap B_\rho| \leq \gamma\rho \|Du\|([u(\cdot, t) > k] \cap B_\rho).$$

Integrating in dt over the time interval $(-\xi\omega\rho, 0]$, gives

$$\frac{|[u < \mu_- + \xi\omega] \cap Q_\rho(\xi\omega)|}{|Q_\rho(\xi\omega)|} \leq \frac{\gamma}{(\xi\omega_o)^2} \frac{\rho}{|Q_\rho|} \int_{-\rho}^0 \|Du(\cdot, t)\|(B_\rho) dt.$$

By the assumption, the right-hand side tends to zero as $\rho \searrow 0$. Hence, there exists ρ so small that

$$\frac{|[u < \mu_- + \xi\omega] \cap Q_\rho(\xi\omega)|}{|Q_\rho(\xi\omega)|} \leq \nu_-$$

where ν_- is the number claimed by Lemma 6.1 for such choice of parameters. The Lemma then implies

$$\operatorname{ess\,inf}_{Q_{\frac{1}{2}\rho}(\xi\omega)} u \geq \mu_- + \frac{1}{2}\xi\omega,$$

and hence

$$\operatorname{ess\,osc}_{Q_{\frac{1}{2}\rho}(\xi\omega)} u \leq \eta\omega \quad \text{where} \quad \eta = 1 - \frac{1}{2}\xi \in (0, 1).$$

Setting $\rho_1 = \frac{1}{2}\xi\omega\rho$ gives

$$\omega_{\rho_1} = \operatorname{ess\,osc}_{Q_{\rho_1}} u \leq \eta\omega.$$

Repeat now the same argument starting from the cylinder Q_{ρ_1} , and proceed recursively to generate a decreasing sequence of radii $\{\rho_n\} \rightarrow 0$ such that

$$\omega_o \leq \operatorname{ess\,osc}_{Q_{\rho_n}} u \leq \eta^n \omega \quad \text{for all } n \in \mathbb{N}. \quad \blacksquare$$

Appendix A Proof of Proposition 3.1

The proof uses an approximation procedure of [2]. Observe first that the assumption $u_t \in L^1_{\text{loc}}(E_T)$ permits to cast (1.1) in the form

$$\|Du(t)\|(E) \leq \|D(u + \varphi)(t)\|(E) - \int_E u_t \varphi dx \quad (\text{A.1})$$

for a.e. $t \in (0, T)$ for all

$$\varphi \in BV_{\text{loc}}(E) \cap L^\infty_{\text{loc}}(E) \quad \text{with} \quad \operatorname{supp}\{\varphi\} \subset E. \quad (\text{A.2})$$

We only prove the estimate for $(u - k)_+$, the one for $(u - k)_-$ being similar. Fix a cylinder

$$[(x_o, t_o) + Q_\rho(\theta)] \subset E_T.$$

Up to a translation, assume that $(x_o, t_o) = (0, 0)$ and fix a time $t \in (-\theta\rho, 0)$ for which

$$\int_{B_\rho} |u_t(x, t)| dx < \infty, \quad \text{and} \quad u(\cdot, t) \in BV(E) \cap L^\infty(B_\rho).$$

The next approximation procedure is carried out for such t fixed and we write $u(\cdot, t) = u$. By [9, Theorem 1.17], there exists $\{u_j\} \subset C^\infty(B_\rho)$ such that

$$\lim_{j \rightarrow \infty} \int_{B_\rho} |u_j - u| dx = 0 \quad \text{and} \quad \|Du\|(E) = \lim_{j \rightarrow \infty} \int_E |Du_j| dx. \quad (\text{A.3})$$

Test (A.1) with $\varphi = -\zeta(u - k)_+$, where $\zeta \in \mathcal{C}(Q_\rho(\theta))$. This is an admissible choice, since $u \in BV(E) \cap L^\infty(B_\rho)$. Set $\varphi_j = -\zeta(u_j - k)_+$ for $j \in \mathbb{N}$. For a given $\epsilon > 0$ there exists $j_o \in \mathbb{N}$ such that

$$\int_E |Du_j| dx < \|Du(\cdot, t)\|(E) + \frac{1}{2}\epsilon \quad \text{for all } j \geq j_o.$$

Here we have used the second of (A.3). By the first, $\{(u_j + \varphi_j)\} \rightarrow (u + \varphi)$ in $L^1(E)$. Therefore, for any $\boldsymbol{\psi} \in [C_o^1(E)]^N$ with $\|\boldsymbol{\psi}\| \leq 1$,

$$\begin{aligned} \int_E (u + \varphi) \operatorname{div} \boldsymbol{\psi} \, dx &= \lim_{j \rightarrow \infty} \int_E (u_j + \varphi_j) \operatorname{div} \boldsymbol{\psi} \, dx \\ &\leq \liminf_{j \rightarrow \infty} \int_E |D(u_j + \varphi_j)| \, dx. \end{aligned}$$

Taking the supremum over all such $\boldsymbol{\psi}$ gives

$$\|D(u + \varphi)(t)\|(E) \leq \liminf_{j \rightarrow \infty} \int_E |D(u_j + \varphi_j)| \, dx.$$

Therefore, up to redefining j_o we may also assume that

$$\int_E |D(u_j + \varphi_j)| \, dx \geq \|D(u + \varphi)\|(E) - \frac{1}{2}\epsilon \quad \text{for all } j \geq j_o.$$

Combining the preceding inequalities gives that

$$\begin{aligned} \int_E |Du_j| \, dx &< \|Du(\cdot, t)\|(E) + \frac{1}{2}\epsilon \\ &\leq \|D(u + \varphi)(\cdot, t)\|(E) + \int_E u_t(\cdot, t) \varphi \, dx + \frac{1}{2}\epsilon \\ &\leq \int_E |D(u_j + \varphi_j)| \, dx + \int_E u_t(\cdot, t) \varphi \, dx + \epsilon \end{aligned} \tag{A.4}$$

for all $j \geq j_o$. Next, estimate the first integral on the right-hand side as,

$$\begin{aligned} \int_E |D(u_j + \varphi_j)| \, dx &= \int_E |D(u_j - \zeta(u_j - k)_+)| \, dx \\ &\leq \int_E |Du_j - \zeta D(u_j - k)_+| \, dx + \int_E |D\zeta|(u_j - k)_+ \, dx \\ &\leq \int_E (1 - \zeta) |Du_j| + \zeta |Du_j - D(u_j - k)_+| \, dx + \int_E |D\zeta|(u_j - k)_+ \, dx. \end{aligned}$$

Put this in (A.4), and absorb the first integral on the right-hand side into the left-hand side, to obtain

$$\begin{aligned} \int_E \zeta |D(u_j - k)_+| \, dx &= \int_E \zeta [|Du_j| - |Du_j - D(u_j - k)_+|] \, dx \\ &\leq \int_E |D\zeta|(u_j - k)_+ \, dx + \int_E u_t(\cdot, t) \varphi \, dx + \epsilon. \end{aligned}$$

From this

$$\int_E |\zeta D(u_j - k)_+| \, dx \leq 2 \int_E |D\zeta|(u_j - k)_+ \, dx + \int_E u_t(\cdot, t) \varphi \, dx + \epsilon.$$

Next let $j \rightarrow \infty$, using the lower semicontinuity of the total variation with respect to L^1 -convergence. This gives

$$\begin{aligned} \|D(\zeta(u-k)_+)\|(B_\rho) &\leq \liminf_{j \rightarrow \infty} \int_E |D(\zeta(u_j-k)_+)| dx \\ &\leq \lim_{j \rightarrow \infty} 2 \int_E |D\zeta|(u_j-k)_+ dx + \int_E u_t \varphi dx + \epsilon \\ &= 2 \int_E |D\zeta|(u-k)_+ dx + \int_E u_t \varphi dx + \epsilon. \end{aligned}$$

Finally let $\epsilon \rightarrow 0$ and use the definition of φ to get

$$\|D(\zeta(u-k)_+)\|(B_\rho) \leq 2 \int_{B_\rho} |D\zeta|(u-k)_+ dx - \int_{B_\rho} \zeta u_t (u-k)_+ dx.$$

To conclude the proof, integrate in dt over $(-\theta\rho, 0)$. ■

Appendix B Boundedness of Minimizers

Proposition B.1 *Let $u : E_T \rightarrow \mathbb{R}$ be a parabolic minimizer of the total variation flow in the sense of (1.1). Furthermore, assume that $u \in L^r_{\text{loc}}(E_T)$ for some $r > N$, and that it can be constructed as the limit in $L^r_{\text{loc}}(E_T)$ of a sequence of parabolic minimizers satisfying (1.2). Then, there exists a positive constant γ depending only upon N, γ, r , such that*

$$\begin{aligned} \sup_{B_\rho(y) \times [s, t]} u_\pm &\leq \gamma \left(\frac{\rho}{t-s} \right)^{\frac{N}{r-N}} \left(\frac{1}{\rho^N(t-s)} \int_{2s-t}^t \int_{B_{4\rho}(y)} u_\pm^r dx d\tau \right)^{\frac{1}{r-N}} \\ &\quad + \gamma \frac{t-s}{\rho} \end{aligned} \tag{B.1}$$

for all cylinders

$$B_{4\rho}(y) \times [s - (t-s), s + (t-s)] \subset E_T.$$

The constant $\gamma(N, \gamma, r) \rightarrow \infty$ as either $r \rightarrow N$, or $r \rightarrow \infty$.

Remark B.1 It is not required that the approximations to u satisfy (1.2) uniformly. The latter is only needed to cast a function satisfying (1.1) into a DeGiorgi class. The proof of the proposition only uses such a membership, and turns such a *qualitative*, non-uniform information into the *quantitative* information (B.1).

Proof (of Proposition B.1). Let $\{u_j\}$ be a sequence of approximating functions to u . Since u_j satisfy (1.2), they belong to the classes $[DG](E_T; 2)$, by Proposition 3.1. It will suffice to establish (B.1) for such u_j for a constant γ independent of j . Thus in the calculations below we drop the suffix j from u_j . The proof will be given for non-negative $u \in [DG]^+(E_T; 2)$, the proof for the remaining

case being identical; it is very similar to the proof of Proposition A.2.1 given in [7, § A.2]. Assume $(y, s) = (0, 0)$ and for fixed $\sigma \in (0, 1)$ and $n = 0, 1, 2, \dots$ set

$$\begin{aligned}\rho_n &= \sigma\rho + \frac{1-\sigma}{2^n}\rho, & t_n &= -\sigma t - \frac{1-\sigma}{2^n}t, \\ B_n &= B_{\rho_n}, & Q_n &= B_n \times (t_n, t).\end{aligned}$$

This is a family of nested and shrinking cylinders with common “vertex” at $(0, t)$, and by construction

$$Q_o = B_\rho \times (-t, t) \quad \text{and} \quad Q_\infty = B_{\sigma\rho} \times (-\sigma t, t).$$

We have assumed that u can be constructed as the limit in $L^r_{\text{loc}}(E_T)$ of a sequence of bounded parabolic minimizers. By working with such approximations, we may assume that u is qualitatively locally bounded. Therefore, set

$$M = \text{ess sup}_{Q_o} \max\{u, 0\}, \quad M_\sigma = \text{ess sup}_{Q_\infty} \max\{u, 0\}.$$

We first find a relationship between M and M_σ . Denote by ζ a non-negative, piecewise smooth cutoff function in Q_n that equals one on Q_{n+1} , and has the form $\zeta(x, t) = \zeta_1(x)\zeta_2(t)$, where

$$\begin{aligned}\zeta_1 &= \begin{cases} 1 & \text{in } B_{n+1} \\ 0 & \text{in } \mathbb{R}^N - B_n \end{cases} & |D\zeta_1| &\leq \frac{2^{n+1}}{(1-\sigma)\rho} \\ \zeta_2 &= \begin{cases} 0 & \text{for } t \leq t_n \\ 1 & \text{for } t \geq t_{n+1} \end{cases} & 0 \leq \zeta_{2,t} &\leq \frac{2^{n+1}}{(1-\sigma)t};\end{aligned}$$

introduce the increasing sequence of levels $k_n = k - 2^{-n}k$, where $k > 0$ is to be chosen, and in $(3.2)_+$, take such a test function, to get

$$\begin{aligned}& \sup_{t_n \leq \tau \leq t} \int_{B_n} [(u - k_{n+1})_+ \zeta]^2(x, \tau) dx + \int_{t_n}^t \|D[(u - k_{n+1})_+ \zeta](\cdot, \tau)\| (B_n) d\tau \\ & \leq \frac{\gamma 2^n}{(1-\sigma)\rho} \iint_{Q_n} (u - k_{n+1})_+ dx d\tau \\ & + \frac{\gamma 2^n}{(1-\sigma)t} \iint_{Q_n} (u - k_{n+1})_+^2 dx d\tau.\end{aligned} \tag{B.2}$$

Estimate

$$\begin{aligned}\iint_{Q_n} (u - k_{n+1})_+ dx d\tau &\leq \gamma \frac{2^{n(r-1)}}{k^{r-1}} \iint_{Q_n} (u - k_n)_+^r dx d\tau, \\ \iint_{Q_n} (u - k_{n+1})_+^2 dx d\tau &\leq \gamma \frac{2^{n(r-2)}}{k^{r-2}} \iint_{Q_n} (u - k_n)_+^r dx d\tau.\end{aligned}$$

Taking these estimates into account yields

$$\begin{aligned} & \sup_{t_n < \tau \leq t} \int_{B_n} [(u - k_{n+1})_+ \zeta]^2(x, \tau) dx + \int_{t_n}^t \|D[(u - k_{n+1})_+ \zeta](\cdot, \tau)\|(B_n) d\tau \\ & \leq \gamma \frac{2^{nr}}{(1-\sigma)t} \left[\left(\frac{t}{\rho} \right) k^{1-r} + \frac{1}{k^{r-2}} \right] \iint_{Q_n} (u - k_n)_+^r dx d\tau. \end{aligned}$$

Assuming that $k > \frac{t}{\rho}$, this implies

$$\begin{aligned} & \sup_{t_n < \tau \leq t} \int_{B_n} [(u - k_{n+1})_+ \zeta]^2(x, \tau) dx + \int_{t_n}^t \|D[(u - k_{n+1})_+ \zeta](\cdot, \tau)\|(B_n) d\tau \\ & \leq \frac{\gamma 2^{nr}}{(1-\sigma)t} \frac{1}{k^{r-2}} \iint_{Q_n} (u - k_n)_+^r dx d\tau. \end{aligned}$$

Set

$$Y_n = \frac{1}{|Q_n|} \iint_{Q_n} (u - k_n)_+^r dx d\tau$$

and estimate

$$Y_{n+1} \leq \|u\|_{\infty, Q_o}^{r-q} \left(\frac{1}{|Q_n|} \iint_{Q_n} (u - k_{n+1})_+^q dx d\tau \right),$$

where $q \stackrel{\text{def}}{=} \frac{N+2}{N}$. Applying the embedding Proposition 4.1 of [7, Preliminaries], the previous inequality can be rewritten as

$$Y_{n+1} \leq \gamma \|u\|_{\infty, Q_o}^{r-q} \left(\frac{\rho}{t} \right) \frac{b^n}{(1-\sigma)^{\frac{1}{N}(N+1)}} \frac{1}{k^{(r-2)\frac{N+1}{N}}} Y_n^{1+\frac{1}{N}},$$

where $b = 2^{r\frac{N+1}{N}}$. Apply Lemma 5.1 of [7, Preliminaries], and conclude that $Y_n \rightarrow 0$ as $n \rightarrow +\infty$, provided k is chosen to satisfy

$$Y_o = \iint_{Q_o} u^r dx d\tau = \gamma (1-\sigma)^{N+1} \|u\|_{\infty, Q_o}^{-(r-q)N} \left(\frac{t}{\rho} \right)^N k^{(r-2)(N+1)},$$

which yields

$$M_\sigma \leq \tilde{\gamma} \frac{M^{\frac{N(r-q)}{(N+1)(r-2)}}}{(1-\sigma)^{\frac{1}{r-2}}} \left(\frac{\rho}{t} \right)^{\frac{N}{(N+1)(r-2)}} \left(\iint_{Q_o} u^r dx d\tau \right)^{\frac{1}{(r-2)(N+1)}}.$$

The proof is concluded by the interpolation Lemma 5.2 of [7, Preliminaries]. ■

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